Profile decomposition and phase control for circle-valued maps in one dimension

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Abstract. When $1 < p < \infty$, maps f in $W^{1/p,p}((0,1);\mathbb{S}^1)$ have $W^{1/p,p}$ phases φ, but the $W^{1/p,p}$-seminorm of φ is not controlled by the one of f. Lack of control is illustrated by "the kink": $f = e^{\imath\varphi}$, where the phase φ moves quickly from 0 to 2π. A similar situation occurs for maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, with Moebius maps playing the role of kinks. We prove that this is the only loss of control mechanism: each map $f : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be written as $f = e^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\pm 1}$, where M_{a_j} is a Moebius map vanishing at $a_j \in \mathbb{D}$, while the integer $K = K(f)$ and the phase ψ are controlled by M. In particular, we have $K \le c_p M$ for some c_p. When $p = 2$, we obtain the sharp value of c_2, which is $c_2 = 1/(4\pi^2)$. As an application, we obtain the existence of minimal maps of degree one in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ with $p \in (2-\varepsilon, 2)$.

Résumé. Décomposition en profils et contrôle des phases des applications unimodulaires en dimension un. Si $1 < p < \infty$, les applications f appartenant à $W^{1/p,p}((0,1);\mathbb{S}^1)$ ont des phases φ dans $W^{1/p,p}$, mais la seminorme $W^{1/p,p}$ de φ n'est pas contrôlée par celle de f. L'absence de contrôle est illustrée par "le pli": $f = e^{\imath\varphi}$, où la phase φ augmente rapidement de 0 à 2π. Pour des applications $f : \mathbb{S}^1 \to \mathbb{S}^1$, le même phénomène apparaît, avec les transformations de Moebius jouant le rôle des plis. Nous prouvons que cet exemple est essentiellement le seul : toute application $f : \mathbb{S}^1 \to \mathbb{S}^1$ telle que $|f|^p_{W^{1/p,p}} \le M$ s'écrit $f = e^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\pm 1}$, où M_{a_j} est une transformation de Moebius s'annulant en $a_j \in \mathbb{D}$, tandis que l'entier $K = K(f)$ et la phase ψ sont contrôlés par M. En particulier, nous avons $K \le c_p M$ pour une constante c_p. Pour $p = 2$, nous obtenons la valeur optimale de c_2, qui est $c_2 = 1/(4\pi^2)$. Comme application, nous obtenons l'existence d'une application minimale de degré un dans $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ avec $p \in]2-\varepsilon, 2[$.

1 Introduction

Let $0 < s < 1$, $1 \le p < \infty$ and let $f : (0,1) \to \mathbb{S}^1$ belong to the space $W^{s,p}$. Then f can be written as $f = e^{\imath\varphi}$, where $\varphi \in W^{s,p}$ [4]. Once the existence of φ is known, a natural question is whether we can control $|\varphi|_{W^{s,p}}$ in terms of $|f|_{W^{s,p}}$. For most of s, p the answer is positive. The exceptional cases are provided precisely by the spaces $W^{1/p,p}((0,1);\mathbb{S}^1)$, with $1 < p < \infty$ [4]. In these spaces, lack of control is established via the following explicit example. For $n \ge 1$, we define φ_n as

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follows:

$$\varphi_n(x) := \begin{cases} 0, & \text{for } 0 < x < 1/2 \\ 2\pi n(x - 1/2), & \text{for } 1/2 < x < 1/2 + 1/n \\ 2\pi, & \text{for } 1/2 + 1/n < x < 1 \end{cases}.$$

Then $|\varphi_n|_{W^{1/p,p}} \to \infty$ (since $\varphi_n \to \varphi = 2\pi\,\chi_{(1/2,1)}$ a.e., and φ does not belong to $W^{1/p,p}$). On the other hand, if we extend $u_n := e^{\imath\varphi_n}$ with the value 1 outside $(0,1)$ and still denote the extension u_n then, by scaling,

$$|u_n|_{W^{1/p,p}((0,1))} \le |u_n|_{W^{1/p,p}(\mathbb{R})} = |u_1|_{W^{1/p,p}(\mathbb{R})} < \infty.$$

Thus $|u_n|_{W^{1/p,p}((0,1))} \lesssim 1$ and $|\varphi_n|_{W^{1/p,p}((0,1))} \to \infty$. Finally, we invoke the fact that $W^{1/p,p}$ phases are unique mod 2π [4].

If one considers instead maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, always in the critical case $f \in W^{1/p,p}$, $1 < p < \infty$, then a new phenomenon occurs: f has a degree $\deg f$, and does not have a $W^{1/p,p}$ phase at all when $\deg f \neq 0$ [11, Remark 10]. However, even if $\deg f = 0$ (and thus f has a $W^{1/p,p}$ phase φ), we have a loss of control phenomenon similar to the one on $(0,1)$. Indeed, let $M_a(z) := \dfrac{a - z}{1 - \overline{a}z}$, $a \in \mathbb{D}$, $z \in \overline{\mathbb{D}}$, be a Moebius transform (that we identify with its restriction to $\mathbb{S}^1$, $M_a : \mathbb{S}^1 \to \mathbb{S}^1$). Let $f_a(z) := \overline{z} M_a(z)$, so that f_a is smooth and $\deg f_a = 0$. One may prove (see below) that $|M_a|_{W^{1/p,p}} = |\mathrm{Id}|_{W^{1/p,p}}$, and thus f_a is bounded in $W^{1/p,p}$. However, if $a \to \alpha = e^{\imath\xi} \in \mathbb{S}^1$, then the smooth phase φ_a of f_a converges a.e. to $\varphi(e^{\imath\theta}) := \begin{cases} \xi - \theta, & \text{if } \xi - \pi < \theta < \xi \\ 2\pi + \xi - \theta, & \text{if } \xi < \theta < \xi + \pi \end{cases}$, which does not belong to $W^{1/p,p}$. [Here, uniqueness of the phases and convergence hold mod 2π.] Thus φ_a is not bounded as $a \to \alpha \in \mathbb{S}^1$. On the other hand, the plot of φ_a shows that φ_a has a "kink shape", and thus we have here the analog of the example on $(0,1)$.

There are evidences that this loss of control mechanism is the only possible one. For example, the phase of the kink is not bounded in $W^{1/p,p}$, but clearly is in $W^{1,1}$ (same for f_a). Bourgain and Brezis [3] proved that for every $f \in W^{1/2,2}((0,1);\mathbb{S}^1)$, we may split $f = e^{\imath\psi} v$, with ψ and $v = e^{\imath\eta}$ satisfying

$$|\psi|_{W^{1/2,2}} \lesssim |f|_{W^{1/2,2}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \lesssim |f|^2_{W^{1/2,2}}. \tag{1}$$

Intuitively, one should think at v as at "the kink part of f". The above result was extended by Nguyen [18] to $1 < p < \infty$: for every $1 < p < \infty$ and every $f \in W^{1/p,p}((0,1);\mathbb{S}^1)$, we may split $f = e^{\imath\psi} v$, with ψ and $v = e^{\imath\eta}$ satisfying

$$|\psi|_{W^{1/p,p}} \le C_p\, |f|_{W^{1/p,p}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \le C_p\, |f|^p_{W^{1/p,p}}. \tag{2}$$

Here we present another result in this direction, written for simplicity on the unit circle.

Theorem 1. *Let $1 < p < \infty$ and $M > 0$. Then there exist constants c_p and $F(M)$ such that: every map $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be written as $f = e^{\imath\psi} \prod_{j=1}^{K} (M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j \in \{-1, 1\}$,*

$$K \le c_p\, M, \tag{3}$$

and

$$|\psi|^p_{W^{1/p,p}} \le F(M). \tag{4}$$

When $p=2$, we may take $c_2 = 1/(4\pi^2)$, and this constant is optimal.

Corollary 1. *Let $1<p<\infty$ and let $f_n, f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ be such that $f_n \rightharpoonup f$ in $W^{1/p,p}$. Then, up to a subsequence, there exist $K \in \mathbb{N}$, $\varepsilon_j \in \{-1,1\}$, $a_{j_n} \in \mathbb{D}$, $\alpha_j \in \mathbb{S}^1$, $j = 1,\dots,K$, $\psi_n \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$, and a constant C, such that:*
i) $f_n = e^{\imath\psi_n}\prod_{j=1}^{K}(M_{a_{j_n}})^{\varepsilon_j} f$;
ii) $a_{j_n} \to \alpha_j$ as $n\to\infty$;
iii) $\psi_n \rightharpoonup C$ in $W^{1/p,p}$ as $n\to\infty$.

The theorem and the corollary are reminiscent of profile decompositions obtained in different, often geometrical, contexts. We mention e.g. the work of Sacks and Uhlenbeck [20] on minimal 2-spheres, the analysis of Brezis and Coron [6, 7, 8] of constant mean curvature surfaces, or the one of Struwe [21] of equations involving the critical Sobolev exponent. There are also abstract approaches to bubbling as in the work of Lions [16] about concentration-compactness or the characterization of lack of compactness of critical embeddings in Gérard [12], Jaffard [15] or Bahouri, Cohen and Koch [1].

Let us comment on the connection between (2) and our theorem. First, (2) has the following version for maps on $\mathbb{S}^1$: we may split $f = e^{\imath\psi}v$, with $|\psi|_{W^{1/p,p}} \le C_p|f|_{W^{1/p,p}}$ and $|v|_{W^{1,1}} \le C_p|f|_{W^{1/p,p}}$. Next, a Moebius maps satisfies $|M_a|_{W^{1,1}} = 2\pi$, and thus

$$\left|\prod_{j=1}^{K}(M_{a_j})^{\varepsilon_j}\right|_{W^{1,1}} \le 2\pi K \le 2\pi\, c_p\, M. \tag{5}$$

Estimate (5) shows that (3) is a refinement of the second part of (2). On the other hand, (4) is weaker than the first part of (2), since $F(M)$ need not have a linear growth (and actually we do not have any control on F). This suggests the following

Conjecture. Let $1<p<\infty$. Then there exist constants c_p, d_p such that every $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be decomposed as $f = e^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j \in \{-1,1\}$,

$$K \le c_p M, \tag{6}$$

and

$$|\psi|^p_{W^{1/p,p}} \le d_p M. \tag{7}$$

In addition, when $p=2$, we may take $c_2 = 1/(4\pi^2)$.

2 Proofs

We start by recalling or establishing few auxiliary results. Given $1 \le p < \infty$, f, f_n will denote maps in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$. When $1<p<\infty$, "$\rightharpoonup$" refers to weak convergence in $W^{1/p,p}$.

1. Recall that, up to a multiplicative factor $\alpha \in \mathbb{S}^1$, the Moebius transforms give all the conformal representations $u : \mathbb{D} \to \mathbb{D}$. In particular, $M_a : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth orientation preserving diffeomorphism, and thus $\deg M_a = 1$. Consequence: if $g : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous, then $\deg[g\circ M_a] = \deg g$.

2. If $1 \le p < \infty$ and $a \in \mathbb{D}$, then $|f\circ M_a|_{W^{1/p,p}} = |f|_{W^{1/p,p}}$. [Here, we let $|f|_{W^{1,1}} := \int_{\mathbb{S}^1}|\dot f| = \int_0^{2\pi}|d[f(e^{\imath\theta})]/d\theta|\,d\theta$ and, for $1<p<\infty$, $|f|^p_{W^{1/p,p}} := \int_{\mathbb{S}^1}\int_{\mathbb{S}^1}|f(x)-f(y)|^p/|x-y|^2\,dxdy$.] In order to prove the desired equality when $p=1$, we write $M_a(e^{\imath\theta}) = e^{\imath\varphi(\theta)}$, $0 \le \theta \le 2\pi$, with φ smooth and increasing. Then

$$|f\circ M_a|_{W^{1,1}} = \int_0^{2\pi}\left|\frac{d}{d\theta}[f(e^{\imath\varphi(\theta)})]\right|d\theta = \int_{\varphi^{-1}(0)}^{\varphi^{-1}(2\pi)}\left|\frac{d}{d\theta}[f(e^{\imath\theta})]\right|d\theta = \int_0^{2\pi}\left|\frac{d}{d\theta}[f(e^{\imath\theta})]\right|d\theta = |f|_{W^{1,1}}.$$

When $1 < p < \infty$, we rely on the following identity, valid for measurable functions $F : \mathbb{S}^1 \times \mathbb{S}^1 \to [0,\infty]$:

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{F(M_a(x), M_a(y))}{|x-y|^2}\, dx dy = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{F(x,y)}{|x-y|^2}\, dx dy. \tag{8}$$

Proof of (8): We have $[M_a]^{-1} = M_a$ and thus, after change of variables, (8) amounts to

$$|x-y|^2\, |\dot{M}_a(x)|\, |\dot{M}_a(y)| = |M_a(x) - M_a(y)|^2, \ \forall\, x, y \in \mathbb{S}^1. \tag{9}$$

In turn, (9) follows immediately from the straightforward equality $|\dot{M}_a(x)| = \dfrac{1-|a|^2}{|1-\overline{a}\, x|^2}$.

3. If $1 \le p < \infty$ and $a \in \mathbb{D}$, then $\deg[f \circ M_a)] = \deg f$. Indeed, to start with, such f has a degree, since $W^{1/p,p} \hookrightarrow$ VMO and VMO maps gave a degree stable with respect to BMO convergence [11]. By item **1**, the desired equality holds true for smooth f. The general case follows by density of $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ into $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ [11, Lemmas A.11 and A.12] and by stability of the VMO degree.

4. If $1 \le p < \infty$ and the degree of f is d, then we may write $f(z) = e^{\imath \psi(z)} z^d$, with $\psi \in W^{1/p,p}(\mathbb{S}^1; \mathbb{R})$. This follows easily from the fact that maps $f \in W^{1/p,p}((0,1); \mathbb{S}^1)$ lift within $W^{1/p,p}$ [4].

5. Let $1 < p < \infty$. For $f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$, let $u = u(f)$ be its harmonic extension. Set $c'_p := \inf\{|f|^p_{W^{1/p,p}};\, u(0) = 0\}$. Clearly, c'_p is achieved, and therefore $c'_p > 0$.

6. When $p = 2$, we have the following straightforward calculations: if $f = \sum_{n\in\mathbb{Z}} a_n\, e^{\imath n\theta}$, then $|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n\in\mathbb{Z}} |n|\, |a_n|^2$ [10, Chapter 13], and $\deg f = \sum_{n\in\mathbb{Z}} n\, |a_n|^2$ [11, eq (25)]. This leads to $4\pi^2\, |\deg f| \le |f|^2_{W^{1/2,2}}$, with equality e.g. when $f(z) := z^d$. On the other hand, if $u(f)(0) = 0$, then $a_0 = 0$ and thus

$$|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n\neq 0} |n|\, |a_n|^2 \ge 4\pi^2 \sum_{n\neq 0} |a_n|^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} |a_n|^2 = 2\pi\, \|f\|^2_{L^2} = 4\pi^2.$$

Thus $c'_2 \ge 4\pi^2$, and the example $f(z) := z$ shows that $c'_2 = 4\pi^2$.

7. For $1 < p < \infty$, there exists some constant c''_p such that $c''_p\, |\deg f| \le |f|^p_{W^{1/p,p}}$, $\forall\, f \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$ [5, Corollary 0.5]. We let c''_p be the best constant such that this estimate holds, and set $c^*_p := \min\{c'_p, c''_p\}$. We also set $c_p := 1/c^*_p$. By item **6**, for $p = 2$ we have $c''_2 = c'_2 = c^*_2 = 4\pi^2$, and $c_2 = 1/(4\pi^2)$.

8. Let $1 < p < \infty$. Let $\delta > 0$ and assume that $|u(f)| \ge \delta$ in $\mathbb{D}$. Then there exists some $C = C(\delta, p)$ such that

$$f = e^{\imath\psi}, \text{ with } \psi \in W^{1/p,p}(\mathbb{S}^1; \mathbb{R}) \text{ and } |\psi|_{W^{1/p,p}} \le C\, |f|_{W^{1/p,p}}. \tag{10}$$

Indeed, set $v := u/|u|$, and write $v = e^{\imath\varphi}$, with smooth φ. By standard properties of the functional calculus and of trace theory, and by the lifting estimates in [4], we have $\varphi \in W^{2/p,p}(\mathbb{D}; \mathbb{R})$, and then $\psi := \operatorname{tr}\varphi \in W^{1/p,p}(\mathbb{S}^1; \mathbb{R})$ satisfies

$$|\psi|_{W^{1/p,p}} \le C(p)\, |\varphi|_{W^{2/p,p}} \le C(p)\, |v|_{W^{2/p,p}} \le C(\delta, p)\, |u|_{W^{2/p,p}} \le C(\delta, p)\, |f|_{W^{1/p,p}}.$$

9. Let $1 < p < \infty$ and $c < c'_p$. If $|f|^p_{W^{1/p,p}} \le c$, then there exists some $\delta > 0$ such that $|u(f)| \ge \delta$ in $\mathbb{D}$. Proof by contradiction: assume that $|f_n|^p_{W^{1/p,p}} \le c$, $f_n \rightharpoonup g$ and and $|u(f_n)(a_n)| \le 1/n$. Since $u(g \circ M_a) = [u(g)] \circ M_a$, we may assume (by item **2**) that $a_n = 0$. We find that $u(f)(0) = 0$ and $|f|^p_{W^{1/p,p}} < c'_p$, which is impossible.

10. Let $1 < p < \infty$. Assume that $f_n \rightharpoonup f$ and $f_n \to f$ a.e. Then $|f_n|^p_{W^{1/p,p}} = |f|^p_{W^{1/p,p}} + |f_n\, \overline{f}|^p_{W^{1/p,p}} + o(1)$. Indeed, if we set $g_n := f_n\, \overline{f}$, then this follows from the Brezis-Lieb lemma [9] and the identity

$$\overline{g_n}(x)[f_n(x) - f_n(y)] = f(x) - f(y) + \overline{g_n}(x) f(y) [g_n(x) - g_n(y)].$$

Proof of the Theorem 1. The proof is by complete induction on the integer part $L := I(c_p M) = I(M/c_p^*)$ of $c_p M$. The case where $L = 0$ follows from items **8** and **9**. Let $L > 0$ and let M be such that $I(M/c_p^*) = L$. Assume, by contradiction, that the theorem does not hold for M. We may thus find a sequence (f_n) with the following properties:
(a) $|f_n|^p_{W^{1/p,p}} \le M$.
(b) For any $K \le L$ and any choice of $a_1,\dots,a_K \in \mathbb{D}$ and of signs $\varepsilon_j = \pm 1$ such that $\sum_{j=1}^K \varepsilon_j = \deg f_n$, if we write $f_n = e^{\imath\psi_n}\prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$, then we have $|\psi_n|_{W^{1/p,p}} \to \infty$. [It is always possible to take K, a_j, ε_j and ψ_n as above: it suffices to let $K := |\deg f| \le I(M/c_p'') \le I(M/c_p^*) = L$, $\varepsilon_j := \operatorname{sgn}\deg f$, and $a_j = 0$.]

By item **8** and property **(b)**, there exist points $a_n \in \mathbb{D}$ such that $u(f_n)(a_n) \to 0$. By item **2**, we may assume in addition that $a_n = 0$. Thus, in addition to **(a)** and **(b)**, we may assume
(c) $f_n \rightharpoonup f$ and $f_n \to f$ a.e., for some f with $u(f)(0) = 0$.

Set $g_n := f_n\overline{f}$. By item **10** and the definition of c_p', we have $|f|^p_{W^{1/p,p}} \ge c_p' \ge c_p^*$, and $|g_n|^p_{W^{1/p,p}} = M - |f|^p_{W^{1/p,p}} + o(1)$. Let $N > M - |f|^p_{W^{1/p,p}}$ be such that $I(N/c_p^*) = I((M - |f|^p_{W^{1/p,p}})/c_p^*) \le L - 1$. For large n, we have $|g_n|^p_{W^{1/p,p}} \le N$. By the induction hypothesis, we may write (possibly up to a subsequence) $g_n = e^{\imath\eta_n}\prod_{j=1}^R (M_{b_{j_n}})^{\varepsilon_j}$, with $|\eta_n|^p_{W^{1/p,p}} \le F(N)$ and $R \le N/c_p^*$. On the other hand, if $d := \deg f$, $b_{j_n} := 0$ and $\varepsilon_j := \operatorname{sgn} d$, then we may write $f = e^{\imath\eta}\prod_{j=R+1}^{R+|d|}(M_{b_{j_n}})^{\varepsilon_j}$, with $\eta \in W^{1/p,p}$ (item **4**). In addition, we have $|d| \le |f|^p_{W^{1/p,p}}/c_p''$ (item **7**). Finally, with $\psi_n := \eta_n + \eta$ and $K := R + |d| \le M/c_p^*$, we have $f_n = e^{\imath\psi_n}\prod_{j=1}^K (M_{b_{j_n}})^{\varepsilon_j}$, and (ψ_n) is bounded in $W^{1/p,p}$. This contradiction completes the proof of the first part of the theorem.

Optimality of (3) when $p = 2$ follows from the fact that, by item **6**, $f(z) := z^d$, $d > 0$, satisfies $|f|^2_{W^{1/2,2}} = c_2 d$ and requires at least d Moebius maps in its decomposition. □

Proof of Corollary 1. By replacing f_n with $f_n\overline{f}$, we may assume that $f_n \rightharpoonup 1$. Up to a subsequence, we may write $f_n = e^{\imath\eta_n}\prod_{j=1}^P (M_{a_{j_n}})^{\varepsilon_j}$, with $a_{j_n} \to \alpha_j \in \overline{\mathbb{D}}$, $j = 1,\dots,P$, and $\eta_n \rightharpoonup \eta$. With no loss of generality, we assume that $\alpha_1,\dots,\alpha_K \in \mathbb{S}^1$ and $\alpha_{K+1},\dots,\alpha_P \in \mathbb{D}$. Since (clearly) $M_{a_{j_n}} \rightharpoonup \alpha_j$, $j = 1,\dots,K$, we find that $1 = e^{\imath(\eta - C)}\prod_{j=K+1}^P (M_{\alpha_j})^{\varepsilon_j}$ for some appropriate C. Thus, with $\zeta_n := \eta_n - \eta$, we have

$$f_n = e^{\imath(\zeta_n + C)}\prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j}\prod_{j=K+1}^P \left(M_{a_{j_n}} M_{\alpha_j}^{-1}\right)^{\varepsilon_j} = e^{\imath\psi_n}\prod_{j=1}^K (M_{a_{j_n}})^{\varepsilon_j},$$

for some ψ_n such that $\psi_n - \zeta_n \to C$ in $W^{1/p,p}$, and thus $\psi_n \rightharpoonup C$. □

3 Applications

We start with an immediate consequence of Theorem 1.

Corollary 2. *Let d be a non negative integer and $\delta > 0$. Then there exist a constant $F(d,\delta)$ such that: every map $f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $\deg f = d$ and $|f|^2_{W^{1/2,2}} \le 4\pi^2(d+1) - \delta$ dans be written as $f = e^{\imath\psi}\prod_{j=1}^d M_{a_j}$, with $|\psi|^2_{W^{1/2,2}} \le F(d,\delta)$.*

Corollary 2 with $d = 1$, as well as a weak version of the corollary when $d \ge 2$ were obtained in [2, Theorem 4.4, Theorem 4.8]. As an application of Corollary 2, we obtain

Theorem 2. *There exists some $\varepsilon > 0$ such that, for $p \in (2-\varepsilon, 2]$,*

$$m_p := \min\{|f|^p_{W^{1/p,p}}; \deg f = 1\}$$

is achieved.

Proof. When $p = 2$, it follows from item **6** that m_2 is achieved by multiples of Moebius maps.

When $1 < p < 2$, consider a minimizing sequence for m_p. Since $m_p \le |\mathrm{Id}|^p_{W^{1/p,p}} := I_p$, we may assume that

$$|f_n|^p_{W^{1/p,p}} \le I_p \to I_2 = 4\pi^2 \text{ as } p \to 2. \tag{11}$$

On the other hand, when $f : \mathbb{S}^1 \to \mathbb{S}^1$ we have $|f|^2_{H^{1/2}} \le 2^{2-p}|f|^p_{W^{1/p,p}}$. Thus

$$|f_n|^2_{H^{1/2}} \le J_p := 2^{2-p} I_p \to 4\pi^2 \text{ as } p \to 2. \tag{12}$$

For p sufficiently close to 2 and fixed $\delta > 0$, we have $J_p \le 8\pi^2 - \delta$. We next apply Corollary 2 to f_n and write $f_n = e^{\imath\psi_n} M_{a_n}$, with $|\psi_n|_{W^{1/2,2}} \le F(1,\delta)$. Set $g_n := f_n \circ M_{a_n}$. By item **2**, (g_n) is a minimizing sequence for m_p. On the other hand, we have $g_n = e^{\imath\varphi_n}\,\mathrm{Id}$, with $\varphi_n := \psi_n \circ M_{a_n}$ bounded in $W^{1/2,2}(\mathbb{S}^1;\mathbb{R})$ (by (8)). Therefore, up to a subsequence $\varphi_n \rightharpoonup \varphi$ in $W^{1/2,2}$, and thus $g_n \rightharpoonup g := e^{\imath\varphi}\,\mathrm{Id}$ in $W^{1/2,2}$. We find that $\deg g = 1$. Since (g_n) is bounded in $W^{1/p,p}$, we obtain that $g_n \rightharpoonup g$ in $W^{1/p,p}$. By a standard argument, g achieves m_p. □

Corollary 1 implies the "bubbling-off of circles along a sequence of graphs", in a sense that will be specified below. A basic object within the theory of Cartesian currents of Giaquinta, Modica and Souček [13] is the one of graphs of maps, considered as currents. For smooth maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, the graph is defined (as a current) as follows. Every smooth 1-form γ on $\mathbb{S}^1 \times \mathbb{S}^1$ can be written (uniquely) as

$$\gamma(s,t) = F(s,t)\,\omega(s) + G(s,t)\,\lambda(t);$$

here, ω and λ are the 1-forms given by

$$\omega(s,t) = s_1\,ds_2 - s_2\,ds_1, \text{ respectively } \lambda(t) = t_1\,dt_2 - t_2\,dt_1,\ \forall\, s,t \in \mathbb{S}^1,$$

and F, G are smooth functions. Then (as an oriented curve on $\mathbb{S}^1 \times \mathbb{S}^1$) the graph $\mathscr{G}_f$ of f acts on γ through the formula

$$\langle \mathscr{G}_f, \gamma\rangle = \int_{\mathbb{S}^1} F(s,f(s)) + \int_{\mathbb{S}^1} G(s,f(s))\, f \wedge \partial_\tau f. \tag{13}$$

Clearly, when f is smooth formula (13) defines a current $\mathscr{G}_f \in \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1) := (\Omega^1(\mathbb{S}^1 \times \mathbb{S}^1))^*$. It was proved in [17, Section 3] (see also [14] for a higher dimensional context) that (13) can be used in order to define $\mathscr{G}_f$ as a current when f is merely $W^{1/2,2}$. The key observation is that the integral $\int_{\mathbb{S}^1} G(s,f(s))\, f \wedge \partial_\tau f$ can be interpreted as a duality bracket between $G(\cdot,f) \in W^{1/2,2}$ and $f \wedge \partial_\tau f \in (W^{1/2,2})^*$. If we set

$$\langle \mathscr{G}_f, \gamma\rangle := \int_{\mathbb{S}^1} F(s,f(s)) + \langle G(\cdot,f), f \wedge \partial_\tau f\rangle_{W^{1/2,2},(W^{1/2,2})^*},\ \forall\, f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1), \tag{14}$$

then we obtain a current which coincides with the usual graph of f when f is smooth, and is continuous with respect to the strong $W^{1/2,2}$ convergence [17].

One of the aims of the theory of Cartesian currents is to describe the limiting behavior of graphs under *weak* convergence of maps. In this direction, the following result was obtained in [17, Proposition 3.1].

Proposition 1. *If $f_n \rightharpoonup f$ in $W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$ then, up to a subsequence, there are finitely many points $a_1,\dots,a_m \in \mathbb{S}^1$ and nonzero integers $d_1,\dots,d_m$ such that*

$$\mathscr{G}_{f_n} \to \mathscr{G}_f + \sum_{j=1}^m d_j\,\delta_{a_j} \times [\![\mathbb{S}^1]\!] \text{ in } \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1). \tag{15}$$

Here,

$$\langle \delta_\alpha \times [\![\mathbb{S}^1]\!], \gamma\rangle = \int_{\mathbb{S}^1} G(\alpha, t).$$

A. Pisante [19] showed me that it is still possible to define $\mathscr{G}_f$ and to extend Proposition 1 to maps $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ with $1 < p < \infty$.

Proposition 2. *([19]) Let $1 < p < \infty$. It is possible to define $\mathscr{G}_f \in \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1)$, $\forall\, f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$. This definition is unique and natural, in the following sense:*

1. *$\mathscr{G}_f$ coincides with the usual graph when f is smooth.*
2. *If $f_n \to f$ strongly in $W^{1/p,p}$, then $\mathscr{G}_{f_n} \to \mathscr{G}_f$ in $\mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1)$.*

In particular, the density of $C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ into $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ implies that $\mathscr{G}_f$ is uniquely defined the properties *1* and *2* above.

Proof. Since we use arguments partly similar to the ones in [17, Proof of Proposition 3.1], we do not give all details. Given $G \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$, set

$$g(s) := \fint_{\mathbb{S}^1} G(s,t)\, d\ell(t).$$

Then there exists some $h \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$, that we may choose (at least locally around some fixed G_0) to depend smoothly on G such that

$$G(s,t)\lambda(t) = g(s)\lambda(t) + d_t h(s,t).$$

Here, d_t stands for the partial differential $\partial_{\tau(t)} h(s,t)\,dt$; d_s is defined similarly.

When f is smooth we have

$$\langle \mathscr{G}_f, \gamma\rangle = \int_{\mathbb{S}^1} F(s, f(s)) - \int_{\mathbb{S}^1} d_s h(s, f(s)) + \int_{\mathbb{S}^1} g(s) f(s) \wedge \partial_\tau f(s). \tag{16}$$

Let $d := \deg f$ and write, as in item **4**, $f(z) = e^{\imath\psi(z)} z^d$, with ψ smooth. Then (16) becomes

$$\langle \mathscr{G}_f, \gamma\rangle = \int_{\mathbb{S}^1} F(s, f(s)) - \int_{\mathbb{S}^1} d_s h(s, f(s)) + d\int_{\mathbb{S}^1} g(s) - \int_{\mathbb{S}^1} \partial_\tau g(s)\psi(s). \tag{17}$$

Clearly, formula (17) still makes sense when f is merely in $W^{1/p,p}$ (and thus ψ is merely $W^{1/p,p}$).

It is easy to see that $\mathscr{G}_f$ defined by (17) is a current, and that its dependence on f is continuous. [The latter property comes from the fact that the degree d and the phase ψ depend continuously on f [10].] $\square$

For further use, let us note the following identity, obtained by density: if $\tilde{f} \in C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ and $f(z) = e^{\imath\psi(z)} z^d$, then

$$\begin{aligned}\langle \mathscr{G}_{f\tilde{f}}, \gamma\rangle = &\int_{\mathbb{S}^1} F(s, f(s)\tilde{f}(s)) - \int_{\mathbb{S}^1} d_s h(s, f(s)\tilde{f}(s)) + d\int_{\mathbb{S}^1} g(s) - \int_{\mathbb{S}^1} \partial_\tau g(s)\psi(s)\\ &+ \int_{\mathbb{S}^1} g(s)\tilde{f} \wedge \partial_\tau \tilde{f}(s).\end{aligned} \tag{18}$$

Proposition 3. *([19]) Let $1 < p < \infty$. If $f_n \rightharpoonup f$ in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ then, up to a subsequence, there are finitely many points $\alpha_1,\dots,\alpha_m \in \mathbb{S}^1$ and nonzero integers $d_1,\dots,d_m$ such that*

$$\mathscr{G}_{f_n} \to \mathscr{G}_f + \sum_{j=1}^{m} d_j \delta_{a_j} \times [\![\mathbb{S}^1]\!] \text{ in } \mathscr{D}_1(\mathbb{S}^1 \times \mathbb{S}^1). \tag{19}$$

Sketch of proof. Let $d := \deg f$ and write $f(z) = e^{\imath\psi(z)} z^d$, with $\psi \in W^{1/p,p}$. We write f_n as in Corollary 1. Set $\widetilde{f_n} := f_n \overline{e^{\imath\psi_n} f}$. Using (18) with f and $\widetilde{f}$ replaced by $e^{\imath\psi_n} f$ and $\widetilde{f_n}$, we easily find that

$$\langle \mathscr{G}_{f_n}, \gamma\rangle = \langle \mathscr{G}_f, \gamma\rangle + \sum_{j=1}^{K} \int_{\mathbb{S}^1} g(s)(M_{a_j}^{\varepsilon_j}) \wedge \partial_\tau (M_{a_j}^{\varepsilon_j})(s) + o(1) \text{ as } n \to \infty. \tag{20}$$

A straightforward calculation shows that

$$\int_{\mathbb{S}^1} g(s)(M_{a_j}^{\varepsilon_j}) \wedge \partial_\tau (M_{a_j}^{\varepsilon_j})(s) \to 2\pi\, \varepsilon_j\, g(\alpha_j) \text{ as } j \to \infty,$$

whence the conclusion of the proposition. □

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References

[1] H. Bahouri, A. Cohen, and G. Koch. A general wavelet-based profile decomposition in the critical embedding of function spaces. *Confluentes Math.*, 3(3):387–411, 2011.

[2] L.V. Berlyand, P. Mironescu, V. Rybalko, and E. Sandier. Minimax critical points in Ginzburg-Landau problems with semi-stiff boundary conditions: existence and bubbling. *Comm. Partial Differential Equations*, 39(5):946–1005, 2014.

[3] J. Bourgain and H. Brezis. On the equation $\operatorname{div} Y = f$ and application to control of phases. *J. Amer. Math. Soc.*, 16(2):393–426 (electronic), 2003.

[4] J. Bourgain, H. Brezis, and P. Mironescu. Lifting in Sobolev spaces. *J. Anal. Math.*, 80:37–86, 2000.

[5] J. Bourgain, H. Brezis, and P. Mironescu. Lifting, degree, and distributional Jacobian revisited. *Comm. Pure Appl. Math.*, 58(4):529–551, 2005.

[6] H. Brezis. Problèmes de convergence dans certaines EDP non linéaires et applications géométriques. In *Goulaouic-Meyer-Schwartz seminar, 1983–1984*, pages Exp. No. 14, 11. École Polytech., Palaiseau, 1984.

[7] H. Brezis and J.-M. Coron. Convergence de solutions de H-systèmes et application aux surfaces à courbure moyenne constante. *C. R. Acad. Sci. Paris Sér. I Math.*, 298(16):389–392, 1984.

[8] H. Brezis and J.-M. Coron. Convergence of solutions of H-systems or how to blow bubbles. *Arch. Rational Mech. Anal.*, 89(1):21–56, 1985.

[9] H. Brezis and E.H. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983.

[10] H. Brezis and P. Mironescu. Sobolev Maps with Values into the Circle, Birkhäuser. In preparation.

[11] H. Brezis and L. Nirenberg. Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.)*, 1(2):197–263, 1995.

[12] P. Gérard. Description du défaut de compacité de l'injection de Sobolev. *ESAIM Control Optim. Calc. Var.*, 3:213–233 (electronic), 1998.

[13] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations. I*, volume 37 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998. Cartesian currents.

[14] M. Giaquinta, G. Modica, and J. Souček. On sequences of maps into S^1 with equibounded $W^{1/2}$ energies. *Selecta Math. (N.S.)*, 10(3):359–375, 2004.

[15] S. Jaffard. Analysis of the lack of compactness in the critical Sobolev embeddings. *J. Funct. Anal.*, 161(2):384–396, 1999.

[16] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2):45–121, 1985.

[17] P. Mironescu and A. Pisante. A variational problem with lack of compactness for $H^{1/2}(S^1;S^1)$ maps of prescribed degree. *J. Funct. Anal.*, 217(2):249–279, 2004.

[18] H.-M. Nguyen. Inequalities related to liftings and applications. *C. R. Math. Acad. Sci. Paris*, 346:957–962, 2008.

[19] A. Pisante. Personal communication, 2015.

[20] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Ann. of Math. (2)*, 113(1):1–24, 1981.

[21] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187(4):511–517, 1984.